LECTURE 1:

Introduction to algorithmic problem solving

Outline

- Problem solving
- What is an algorithm ?
- Properties an algorithm should have
- Describing Algorithms
- Types of data to use
- Basic operations

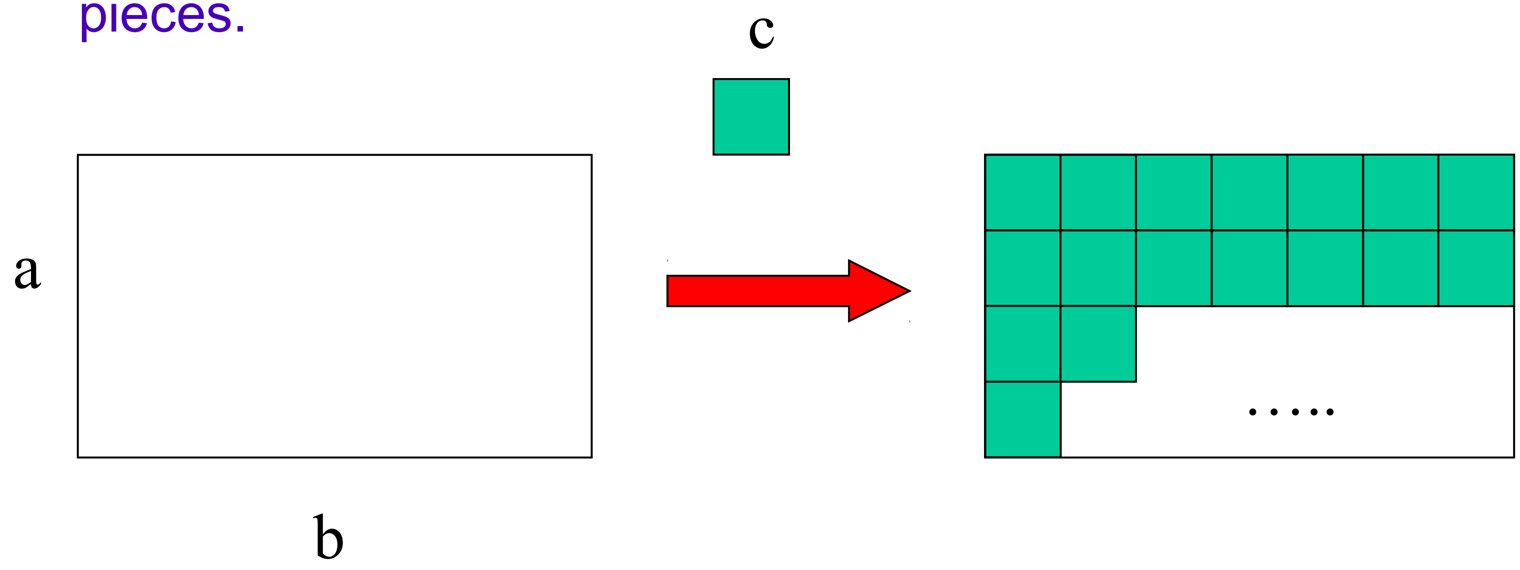
Problem solving

Example:

Consider a rectangle of (integer) size axb.

We want to cover the rectangle with identical square pieces.

Choose the square size that leads to using the smallest number of pieces.



Problem solving

Mathematical statement:

Let a and b be two non-zero natural numbers. Find the natural number c having the following properties:

- c divides a and b (c is a common divisor of a and b)
- c is greater than any other common divisor of a and b

Problem universe: natural numbers (a and b represent the input data, c represents the result)

Problem statement (relations between the input data and the result):

find c, the **greatest common divisor** of **a** and **b**

Problem solving

Remark:

- Previous problem: compute the value of a function (that which associates to a pair of natural numbers the greatest common divisor)

- Another kind of problems: decide if input data satisfies or not certain properties.

  Example: verify if a natural number is a prime number or not

In both cases: want a method which after a finite number of (rather simple) operations produces the solution … such a method is an algorithm

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What is an algorithm ?

Different definitions …

Algorithm = something like a cooking recipe used to solve problems

Algorithm = a step by step problem solving method

Algorithm = a finite sequence of operations applied to some input data in order to obtain the solution of the problem

What is the origin of the word ?

Abū ʿAbdallāh Muḥammad ibn Mūsā

al-Khowarizmi - Persian mathematician (790-840)

↓

algorism ⟶ algorithm

- He used 0 as a digit for the first time
- He wrote the first book on algebra

Examples

Euclid
(325 -265 b.C.)

Algorithms in day by day life:

- using a phone:
  - pick up the phone
  - dial the number
  - talk ….

Algorithms in mathematics:

- Euclid's algorithm (it is considered to be the first algorithm)
  - find the greatest common divisor of two numbers
- Eratostene's algorithm
  - generate prime numbers in a range
- Horner's algorithm
  - compute the value of a polynomial

From a problem to an algorithm

Problem: compute gcd(a,b) = greatest common divisor of a and b

- Input data
  - a, b - natural values
- Solving method
  - Divide a by b and store the remainder
  - Divide b by the remainder and keep the new remainder
  - Repeat the divisions until a zero remainder is obtained
  - The result will be the last non-zero remainder

Algorithm:

- Variables = abstract entities corresponding to data
  - dividend, divisor, remainder
- Operations

1. Assign to the dividend the value of a and to the divisor the value of b
2. Compute the remainder of the division of the dividend by the divisor
3. Assign to the dividend the value of the previous divisor and to the divisor the previous value of the remainder
4. If the remainder is not zero go to step 2 otherwise output the result (the last nonzero remainder)

From an algorithm to a program

Algorithm:

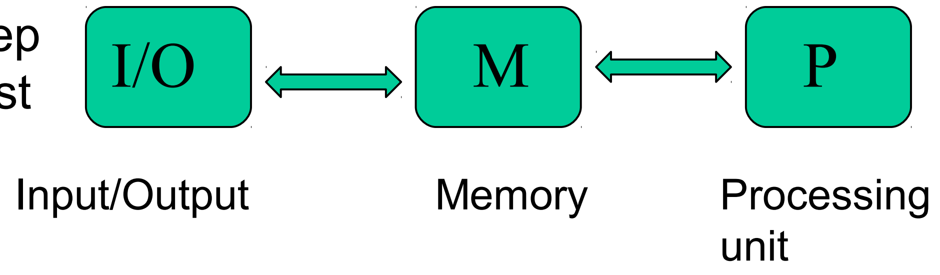
- Variables = abstract entities corresponding to data
  - dividend, divisor, remainder
- Operations

1. Assign to the dividend the value of a and to the divisor the value of b
2. Compute the remainder of the division of the dividend by the divisor
3. Assign to the dividend the value of the previous divisor and to the divisor the previous value of the remainder
4. If the remainder is not zero go to step 2 otherwise output the result (the last nonzero remainder)

Program = alg. description in a programming language**:**

- Variables: each variable has a corresponding storing region in the computer memory
- Operations: each operation can be decomposed in a few actions which can be executed by the computer

Very simplified model of a computer



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Properties an algorithm should have

- Generality
- Finiteness
- Non-ambiguity
- Efficiency

Generality

- The algorithm applies to all instances of input data not only for particular instances

Example:

Let's consider the problem of sorting a sequence of values in increasing order.

For instance:

(2,1,4,3,5) → (1,2,3,4,5)

input data → result

Generality (cont'd)

Method:

Step 1: $\boxed{2 \leftrightarrow 1}$ 4 3 5

Step 2: 1 $\boxed{2 \quad 4}$ 3 5

Step 3: 1 2 $\boxed{4 \leftrightarrow 3}$ 5

Step 4: 1 2 3 $\boxed{4 \quad 5}$

Description:

- compare the first two elements:
  if there are not in the desired order swap them
- compare the second and the third element and do the same

…..

- continue until the last two elements were compared

The sequence has been sorted

Generality (cont'd)

- Is this algorithm a general one ? Does it order ANY sequence of values ?

- Answer: NO

Counterexample:

3 2 1 4 5
2 3 1 4 5
2 1 3 4 5
2 1 3 4 5

The method doesn't work, it isn't a general sorting algorithm

Properties an algorithm should have

- Generality
- Finiteness (halting)
- Non-ambiguity
- Efficiency

Finiteness

- An algorithm have to terminate, i.e. to stop after a finite number of steps

Example

Step1: Assign 1 to x;

Step2: Increase x by 2;

Step3: If x=10 then STOP;
else GO TO Step 2

How does this algorithm work ?

Finiteness (cont'd)

How does this algorithm work and what does it produce?

Step1: Assign 1 to x;

Step2: Increase x by 2;

Step3: If x=10

then STOP;

else Print x; GO TO Step 2;

x=1

x=3 x=5 x=7 x=9 x=11

What can we say about this algorithm ?

The algorithm generates odd numbers but it never stops !

Finiteness (cont'd)

The algorithm which generate all odd naturals in the set {1,2,…,10}:

Step 1: Assign 1 to x;

Step 2: Increase x by 2;

Step 3: If x>=10

then STOP;

else Print x; GO TO Step 2

What properties should an algorithm have ?

- Generality
- Finiteness
- Non-ambiguity
- Efficiency

Non-ambiguity

The operations in an algorithm must be rigorously specified:

- At the execution of each step we have to know exactly which step will be executed next.

Example:

Step 1: Set x to value 0

Step 2: Either increment x by 1 or decrement x by 1

Step 3: If $x \in [-2,2]$ then go to Step 2; else Stop.

As long as we don't have a criterion for deciding whether x is incremented or decremented, the sequence above is not an algorithm.

Non-ambiguity (cont'd)

Modify the previous algorithm as follows:

Step 1: Set x to value 0

Step 2: Flip a coin

Step 3: coins == head

    then increment x by 1

    else decrement x by 1

Step 3: If $x \in [-2,2]$ then go to Step 2, else Stop.

- This time the algorithm can be executed but … different executions may lead to different results
- This is a so called randomized algorithm

Properties an algorithm should have

- Generality
- Finiteness
- Non-ambiguity
- Efficiency

Efficiency

An algorithm should use a reasonable amount of computing resources: memory and time

Finiteness is not enough if we have to wait too much to obtain the result

Example:

Consider a dictionary containing 50000 words.

Write an algorithm that takes a word as input and returns all anagrams of that word appearing in the dictionary.

Example of anagram: ship -> hips

Efficiency

First approach:

Step 1: generate all anagrams of the word

Step 2: for each anagram search for it in the dictionary (using binary search)

Let's consider that:

- the dictionary contains n words
- the analyzed word contains m letters

Rough estimate of the number of basic operations:

- number of anagrams: $m!$
- words comparisons for each anagram: $\log_2 n$ (e.g. binary search)
- letters comparisons for each word: m

$$\boxed{m! * m * \log_2 n}$$

Efficiency

Second approach:

Step 1: sort the letters of the initial word

Step 2: for each word in the dictionary having m letters:

- Sort the letters of this word
- Compare the sorted version of the word with the sorted version of the original word

Rough estimate of the number of basic operations:

- Sorting the initial word needs almost m^2 operations (e.g. insertion sort)
- Sequentially searching the dictionary and sorting each word of length m needs at most nm^2 comparisons
- Comparing the sorted words requires at most nm comparisons

$$\boxed{n\,m^2 + nm + m^2}$$

Efficiency

Which approach is better ?

First approach

$m!\, m \log_2 n$

Second approach

$n\, m^2 + n\, m + m^2$

Example: m=12 (e.g. word algorithmics)
n=50000 (number of words in dictionary)

8* 10^10

8*10^6

one basic operation (e.g.comparison)= 1ms=10^{-3} s

24000 hours

2 hours

Thus, it is important to analyze the efficiency of the algorithm and choose more efficient algorithms

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- Problem solving
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- Basic operations

How can we describe algorithms ?

Solving problems can usually be described in mathematical language

Not always adequate to describe algorithms because:

- Operations which seem elementary when described in a mathematical language are not elementary when they have to be encoded in a programming language

Example: computing a sum, computing the value of a polynomial

Mathematical description

$$\sum_{i=1}^{n} i = 1 + 2 + \ldots + n$$

Algorithmic description

(it should be a sequence of basic operations)

How can we describe algorithms ?

Two basic instruments:

- Flowcharts:
  - graphical description of the flow of processing steps
  - not used very often, somewhat old-fashioned.
  - however, sometimes useful to describe the overall structure of an application
- Pseudocode:
  - artificial language based on
    - vocabulary (set of keywords)
    - syntax (set of rules used to construct the language's "phrases")
  - not as restrictive as a programming language

Why do we call it pseudocode ?

Because …

- It is similar to a programming language (code)

- Not as rigorous as a programming language (pseudo)

In pseudocode the phrases are:

- Statements or instructions (used to describe processing steps)

- Declarations (used to specify the data)

Types of data

Data = container of information

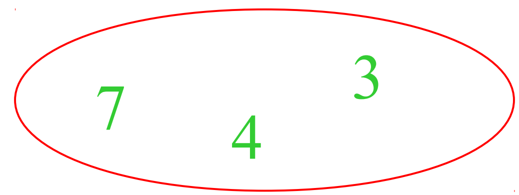
Characteristics:

- name
- value
  - constant (same value during the entire algorithm)
  - variable (the value varies during the algorithm)
- type
  - primitive (numbers, characters, truth values …)
  - structured (arrays)

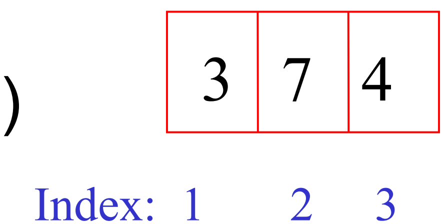
Types of data

Arrays - used to represent:

- Sets (e.g. {3,7,4}={3,4,7})
  - the order of the elements doesn't matter

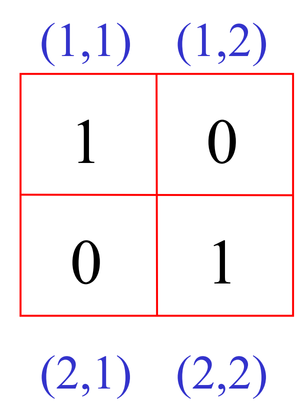


- Sequences (e.g. (3,7,4) is not (3,4,7))
  - the order of the elements matters



- Matrices
  - bidimensional arrays

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



How can we specify data ?

- Simple data:
  - Integers INTEGER <variable>
  - Reals REAL <variable>
  - Boolean BOOLEAN <variable>
  - Characters CHAR <variable>

How can we specify data ?

Arrays

One dimensional

<elements type> <name>[n1..n2]

(ex: REAL x[1..n])

Two-dimensional

<elements type> <name>[m1..m2, n1..n2]

(ex: INTEGER A[1..m,1..n])

How can we specify data ?

Specifying elements:

- One dimensional

  x[i] - i is the element's index

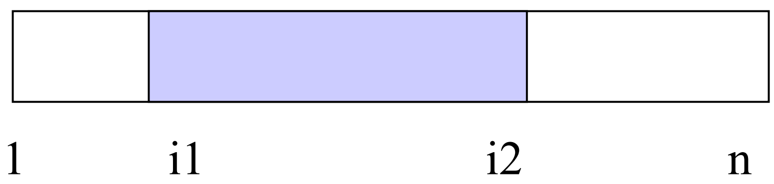
- Two-dimensional

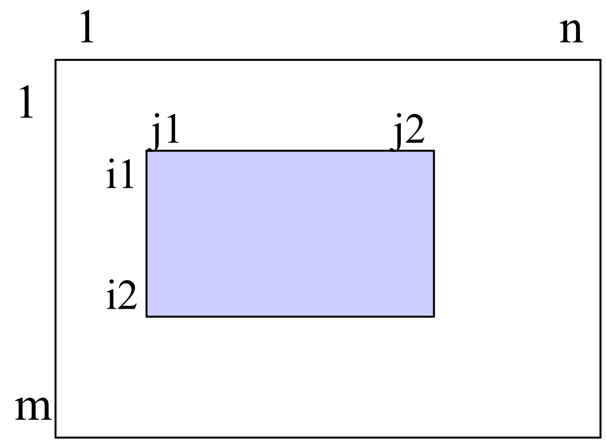
  A[i,j] - i is the row's index, while j is the column's index

How can we specify data ?

Specifying subarrays:

- Subarray= contiguous portion of an array
  - One dimensional: x[i1..i2] (1<=i1<i2<=n)
  - Bi dimensional: A[i1..i2, j1..j2]
    (1<=i1<i2<=m, 1<=j1<j2<=n)





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- Basic instructions

What are the basic instructions ?

Instruction (statement)

= action to be executed by the algorithm

There are two main types of instructions:

- Simple
  - Assignment (assigns a value to a variable)
  - Transfer (reads an input data; writes a result)
  - Control (specifies which is the next step to be executed)
- Structured ….

Assignment

- Aim: give a value to a variable
- Description:

v ← <expression>

Rmk: sometimes we use := instead of ←

- Expression = syntactic construction used to describe a computation

It consists of:

  – Operands: variables, constant values
  – Operators: arithmetical, relational, logical

Operators

- Arithmetical:
  + (addition), - (subtraction), *(multiplication),
  / (division), ^ (power),
  DIV (from divide) or / (integer quotient),
  MOD (from modulo) or % (remainder)

- Relational:
  = (equal), != (different),
  < (less than), <= (less than or equal),
  >(greater than) >= (greater than or equal)

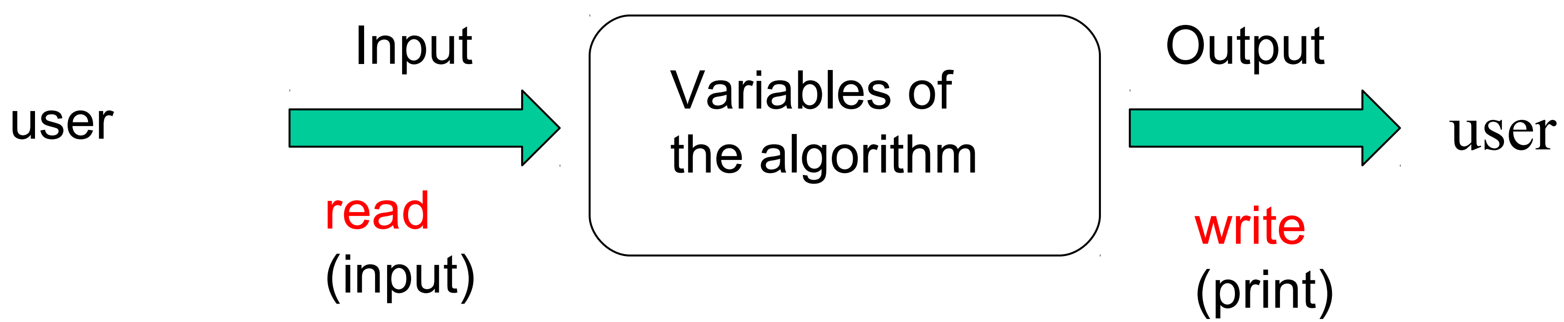
- Logical:
  OR (disjunction), AND (conjunction), NOT (negation)

Input/Output

- Aim:
  - read input data
  - output the results
- Description:

read v1,v2,…   input v1, v2,…

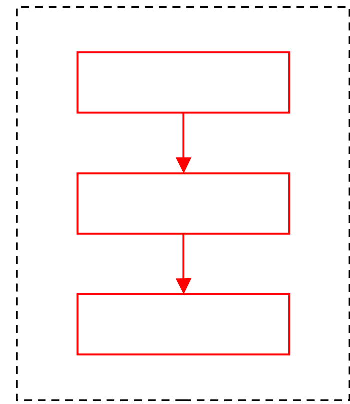
write e1,e2,…   print e1, e2,…



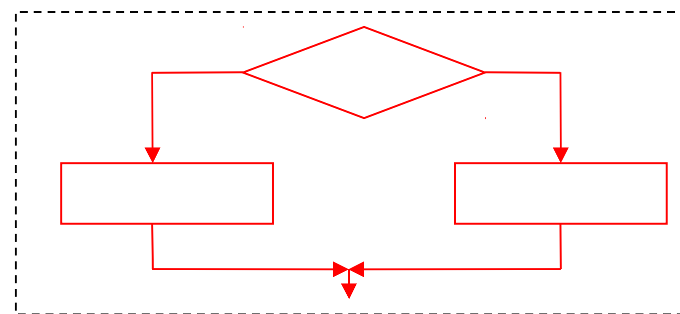
Instructions

Structured:

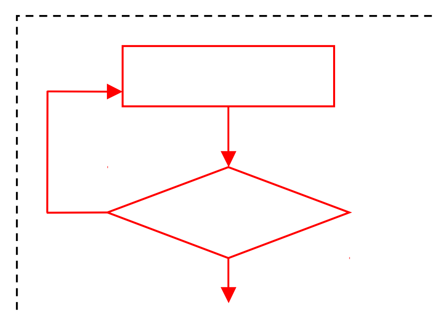
- Sequence of instructions



- Conditional statement

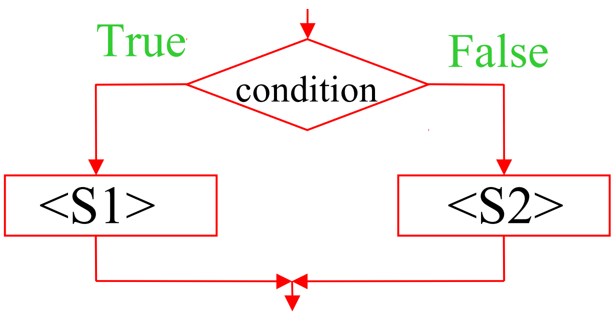


- Loop statement



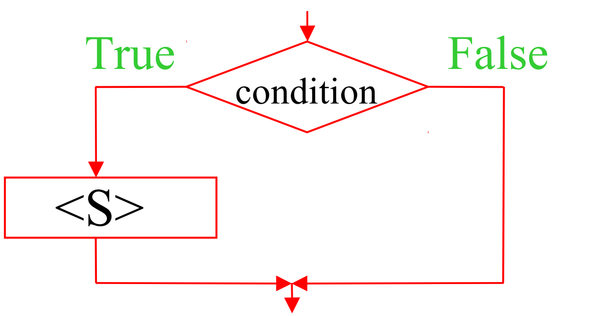
Conditional statement

- Aim: allows choosing between two or several alternatives depending on the value of a/some condition(s)



- General variant:

```
if <condition> then  <S1>
                     else  <S2>
endif
```



- Simplified variant:

```
if <condition> then <S>
endif
```

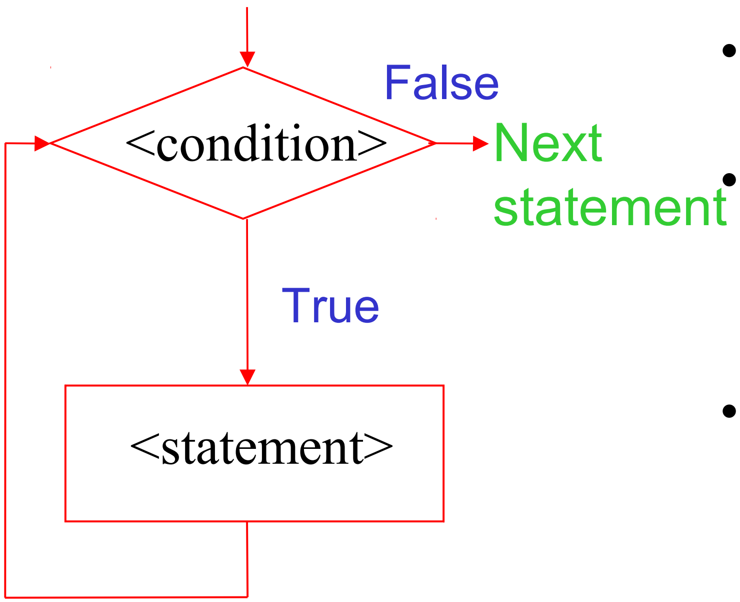
Loop statements

- Aim: allows repeating a processing step
- Example: compute a sum

  S= 1+2+…+i+…+n
- A loop is characterized by:
  - The processing step which have to be repeated
  - A stopping (or continuation) condition

- Depending on the moment of analyzing the stopping condition there are two main loop statements:
  - Preconditioned loops (WHILE loops)
  - Postconditioned loops (REPEAT loops)

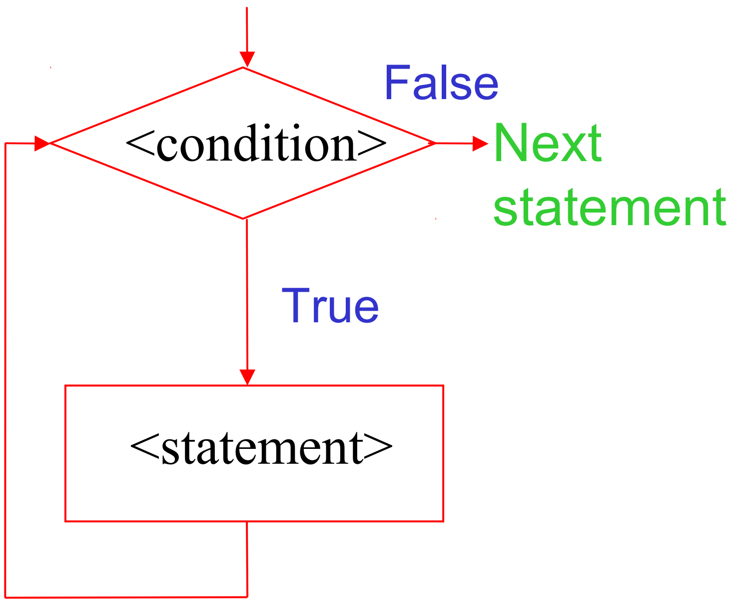
WHILE loop




```
while  <condition> do
       <statement>
endwhile
```

- First, the condition is analyzed
- If it is true then the statement is executed and the condition is analyzed again
- If the condition becomes false the control of execution passes to the next statement in the algorithm
- If the condition never becomes false then the loop is infinite
- If the condition is false from the beginning then the statement inside the loop is never executed

WHILE loop

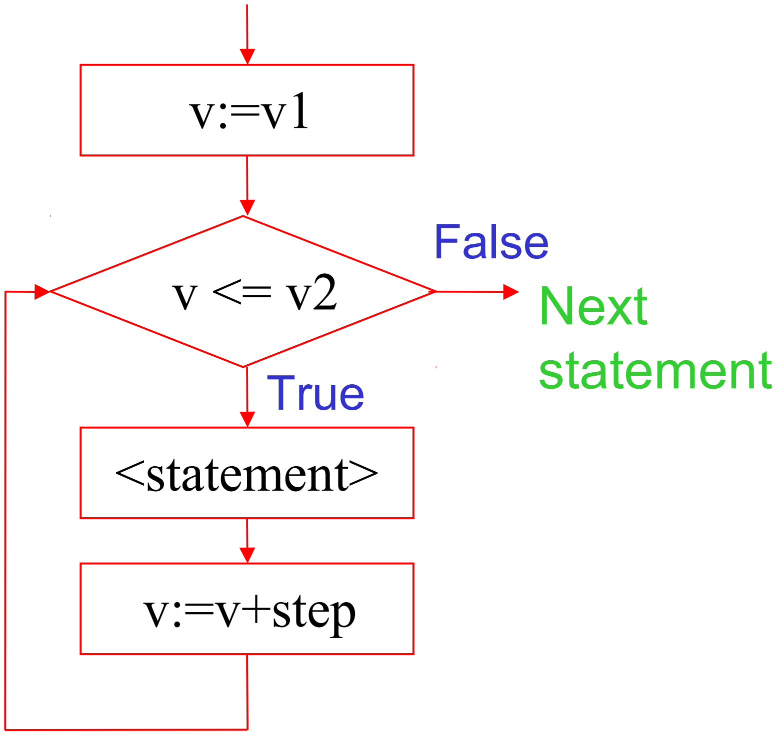


$$\sum_{i=1}^{n} i = 1+2+\ldots+n$$

```
while  <condition> do
       <statement>
endwhile
```

```
S:=0    // initialize the variable which will
        //  contain the result
i:=1    // index intialization
while i<=n do
   S:=S+i  // add the current term to S
    i:=i+1   // prepare the next term
endwhile
```

FOR loop

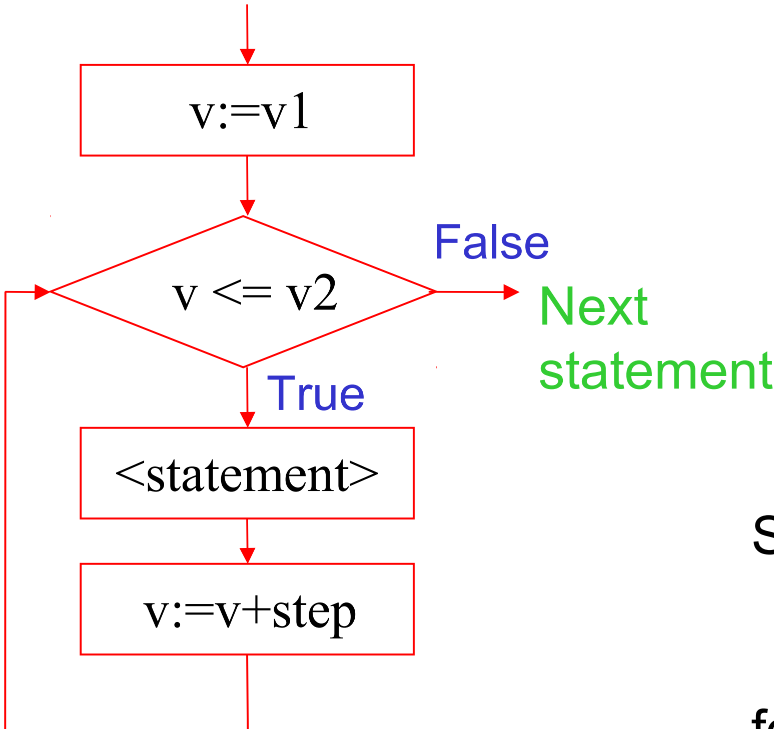


- Sometimes the number of repetitions of a processing step is known apriori
- Then we can use a counting variable which varies from an initial value to a final value using a step value
- Repetitions: v2-v1+1 if step=1

```
for  v:=v1,v2,step do
            <statement>
endfor
```

```
v:=v1
while v<=v2 do
          <statement>
          v:=v+step
endwhile
```

FOR loop



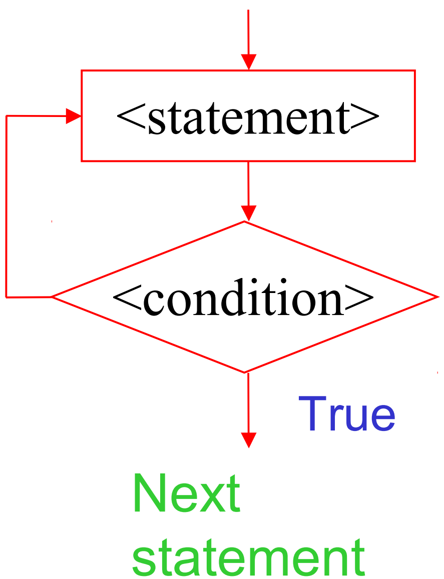
for v:=v1,v2,step do
<statement>
endfor

$$\sum_{i=1}^{n} i = 1 + 2 + \ldots + n$$

```
S:=0   // initialize the variable which will
       //  contain the result

for i:=1,n do
  S:=S+i  // add the term to S
endfor
```

REPEAT loop




```
repeat <statement>
until <condition>
```

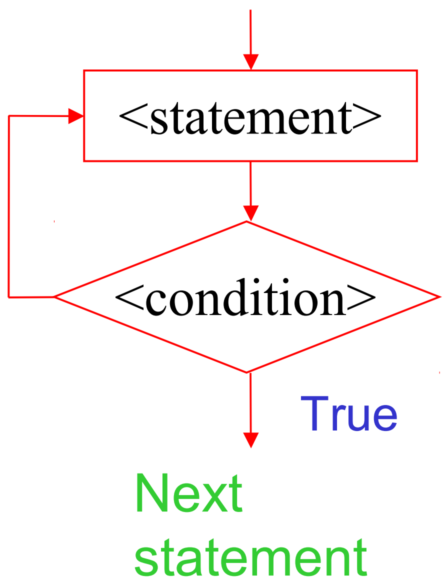
- First, the statement is executed. Thus it is executed at least once

- Then the condition is analyzed and if it is false the statement is executed again

- When the condition becomes true the control passes to the next statement of the algorithm

- If the condition doesn’t become true then the loop is infinite

REPEAT loop



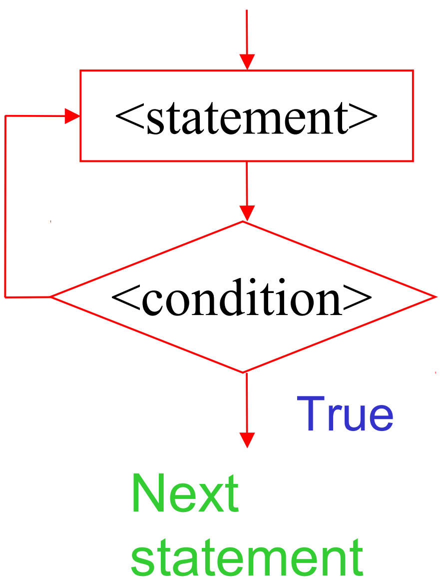
repeat <statement>
until <condition>

$$\sum_{i=1}^{n} i = 1 + 2 + \ldots + n$$

```
S:=0
i:=1
repeat
   S:=S+i
   i:=i+1
until i>n
```

```
S:=0
i:=0
repeat
   i:=i+1
   S:=S+i
until i>=n
```

REPEAT loop



repeat <statement>
until <condition>

Any REPEAT loop can be transformed in a WHILE loop:

<statement>
while NOT <condition> DO
<statement>
endwhile

Summary

- Algorithms are step-by-step procedures for problem solving

- They should have the following properties:
  - Generality
  - Finiteness
  - Non-ambiguity (rigorousness)
  - Efficiency

- Data processed by an algorithm can be
  - simple
  - structured (e.g. arrays)

- We describe algorithms by means of pseudocode

Summary

- Pseudocode:

  Assignment :=

  Data transfer read (input), write (print)

  Decisions if … then … else … endif

  Loops while … do … endwhile
  for … do … endfor
  repeat … until

Next lecture …

- Other examples of algorithms
- Subalgorithms
- A word on correctness